Game Theory

Lecture 04

Nash’s Theorem

The Brouwer Fixed Point Theorem

We will use the following to prove Nash's Theorem.

Theorem(Brouwer, 1909) Every continuous function $f : D \mapsto D$ mapping a compact and convex, nonempty subset $D \subseteq \mathbb{R}^m$ to itself has a "fixed point", i.e., there is $x^* \in D$ such that $f(x^*) = x^*$.

Explanation:

- A "continuous" function is intuitively one whose graph has no "jumps". I.e., any "sufficiently little (non-zero) change" in x can change $f(x)$ by at most "as little (non-zero) change as desired".

- For our current purposes, we don't need to know exactly what "compact and convex" means.

(See the appendix of this lecture for definitions.)

We only state the following fact:

Fact The set of profiles $X = X_1 \times \ldots \times X_n$ is a compact and convex subset of R^m.
(Where $m = \Sigma_{i=1}^n m_i$, recalling that $m_i = |S_i|$.)

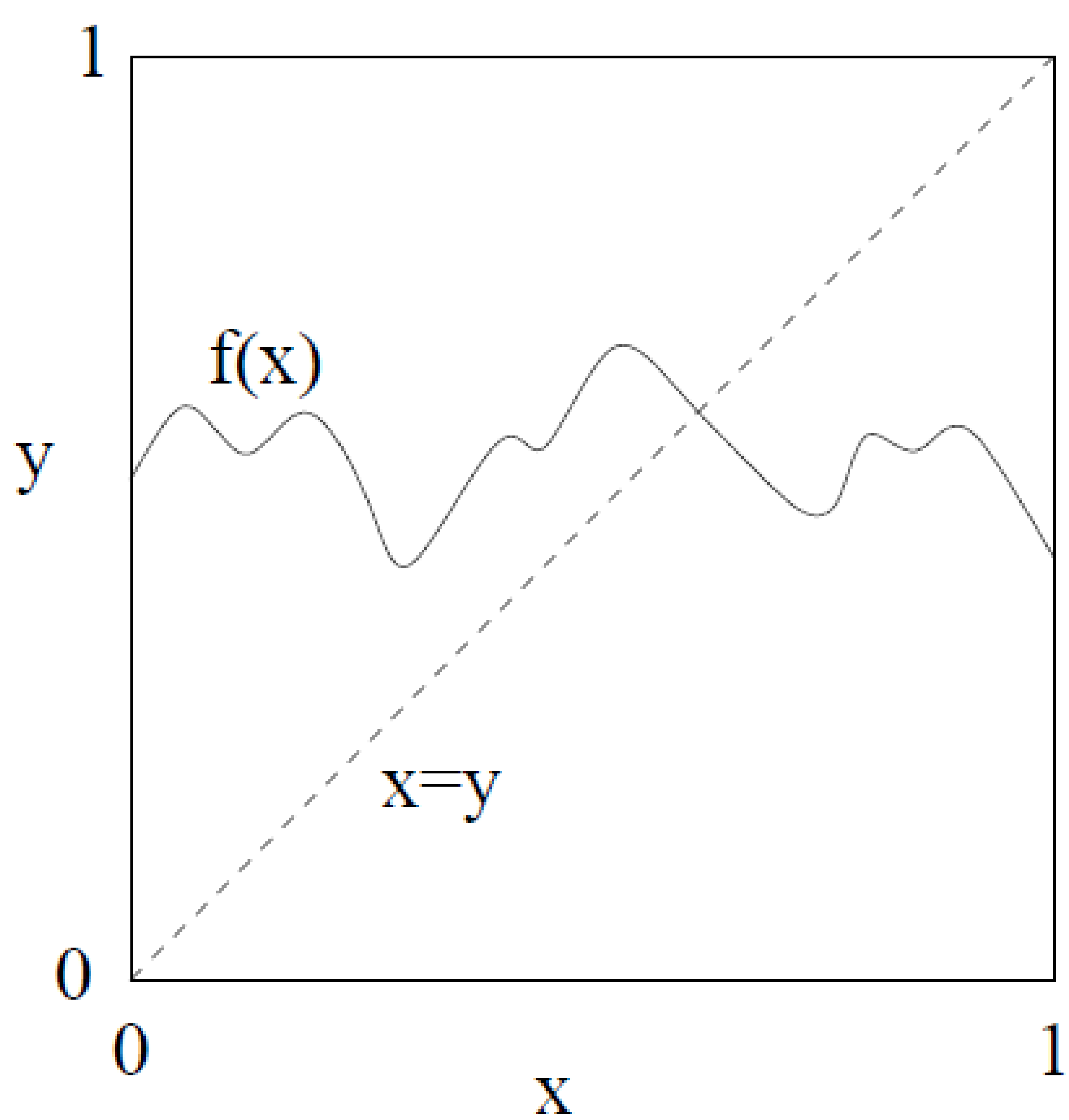
Simple cases of Brouwer's Theorem

To see a simple example of what Brouwer's theorem says, consider the interval $[0,1] = \{x \mid 0 \leq x \leq 1\}$.

$[0,1]$ is compact and convex.
(More generally, $[0,1]^n$ is compact and convex.)

For a continuous $f : [0,1] \mapsto [0,1]$, you can "visualize" why the theorem is true:

The "visual proof" in the 1-dimensional case:



For $f : [0,1]^2 \mapsto [0,1]^2$, the theorem is already far less obvious: "the crumpled sheet experiment".

Some Remarks

- Brouwer's Theorem is a deep and important result in topology.
- It is not very easy to prove, and we won't prove it.
- If you are desperate to see a proof, there are many. See, e.g., any of these:
  - [Milnor'66] (Differential Topology). (uses, e.g., Sard's Theorem).
  - [Scarf'67 & '73, Kuhn'68, Border'89], uses **Sperner's Lemma**.
  - [Rotman'88] (Algebraic Topology). (uses homology, etc.)

Proof of Nash's Theorem

Proof: (Nash's 1951 proof)

We will define a continuous function $f : X \mapsto X$, where $X = X_1 \times \ldots \times X_n$, and we will show that if $f(x^*) = x^*$ then $x^* = (x_1^*, \ldots, x_n^*)$ must be a Nash Equilibrium.

By Brouwer's Theorem, we will be done.

(In fact, it will turn out that x^* is a Nash Equilibrium if and only if $f(x^*) = x^*$.)

recall: A profile $x^* = (x_1^*, \ldots, x_n^*) \in X$ is a Nash Equilibrium if and only if, for every player i, and every pure strategy $\pi_{i,j}$, $j = 1, \ldots, m_i$

$$U_i(x^*) \geq U_i(x_{-i}^*; \pi_{i,j})$$

So, rephrasing our goal, we want to find $x^* = (x_1^*, \ldots, x_n^*)$ such that

$$U_i(x_{-i}^*; \pi_{i,j}) \leq U_i(x^*)$$

i.e., such that

$$U_i(x_{-i}^*; \pi_{i,j}) - U_i(x^*) \leq 0$$

for all players $i \in N$, and all $j = 1, \ldots, m_i$.

For a mixed profile $x = (x_1, x_2, \ldots, x_n) \in X$: let

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\}$$

Intuitively, $\varphi_{i,j}(x)$ measures "how much better off" player i would be if he/she picked $\pi_{i,j}$ instead of x_i (and everyone else remained unchanged).

Define $f : X \mapsto X$ as follows: For $x = (x_1, x_2, \ldots, x_n) \in X$, let

$$f(x) = (x'_1, x'_2, \ldots, x'_n)$$

where for all i, and $j = 1, \ldots, m_i$,

$$x'_i(j) = \frac{x_i(j) + \varphi_{i,j}(x)}{1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x)}$$

Facts:

1. If $x \in X$, then $f(x) = (x'_1, \ldots, x'_n) \in X$.

2. $f : X \mapsto X$ is continuous.

(These facts are not hard to check.)

Thus, by Brouwer, there exists $x^* = (x^*_1, x^*_2, \ldots, x^*_n) \in X$ such that $f(x^*) = x^*$.

Now we have to show x^* is a NE.

For each i, and for $j = 1, \ldots, m_i$,

$$x_i^*(j) = \frac{x_i^*(j) + \varphi_{i,j}(x^*)}{1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)}$$

thus,

$$x_i^*(j)(1 + \sum_{k=1}^{m_i} \varphi_{i,k}(x^*)) = x_i^*(j) + \varphi_{i,j}(x^*)$$

hence,

$$x_i^*(j) \sum_{k=1}^{m_i} \varphi_{i,k}(x^*) = \varphi_{i,j}(x^*)$$

We will show that in fact this implies $\varphi_{i,j}(x^*)$ must be equal to 0 for all j.

Claim: For any mixed profile x, for each player i, there is some j such that $x_i(j) > 0$ and $\varphi_{i,j}(x) = 0$.

<u>*Proof of claim:*</u> For any $x \in X$,

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\}$$

Since $U_i(x)$ is the "weighted average" of $U_i(x_{-i}; \pi_{i,j})$'s, based on the "weights" in x_i, there must be some j used in x_i, i.e., with $x_i(j) > 0$, such that $U_i(x_{-i}; \pi_{i,j})$ is no more than the weighted average. I.e.,

$$U_i(x_{-i}; \pi_{i,j}) \leq U_i(x)$$

I.e.,

$$U_i(x_{-i}; \pi_{i,j}) - U_i(x) \leq 0$$

Therefore,

$$\varphi_{i,j}(x) = \max\{0, U_i(x_{-i}; \pi_{i,j}) - U_i(x)\} = 0$$

■

Thus, for such a j, $x_i^*(j) > 0$ and

$$x_i^*(j) \sum_{k=1}^{m_1} \varphi_{i,k}(x^*) = 0 = \varphi_{i,j}(x^*)$$

But, since $\varphi_{i,k}(x^*)$'s are all ≥ 0, this means $\varphi_{i,k}(x^*) = 0$ for all $k = 1, \ldots, m_i$. Thus,

For all players i, and for $j = 1, \ldots, m_i$,

$$U_i(x^*) \geq U_i(x^*_{-i}; \pi_{i,j})$$

Q.E.D. (Nash's Theorem)

In fact, since $U_i(x^*)$ is the "weighted average" of $U_i(x^*_{-i}, \pi_{i,j})$'s, we see that

Useful Corollary for Nash Equilibria:
$U_i(x^*) = U_i(x^*_{-i}, \pi_{i,j})$, whenever $x_i^*(j) > 0$.

Rephrased: In a Nash Equilibrium x^*, if $x_i^*(j) > 0$ then $U_i(x^*_{-i}; \pi_{i,j}) = U_i(x^*)$; i.e., each such $\pi_{i,j}$ is itself a "best response" to x^*_{-i}.

This is a subtle but very important point.
It will be useful later when we try to compute NE's.

Remarks

- The proof using Brouwer gives ostensibly no clue how to compute a Nash Equilibrium. It just says it exists!

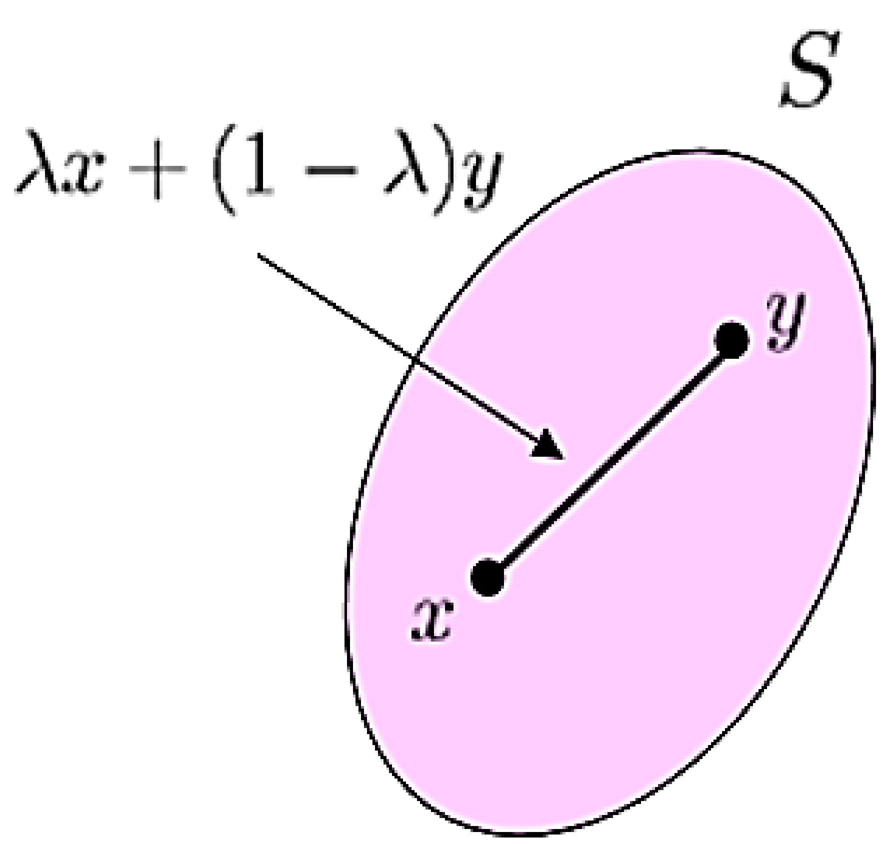
- We will come back to the question of computing Nash Equilibria in general games later in the course.

Appendix: continuity, compactness, convexity

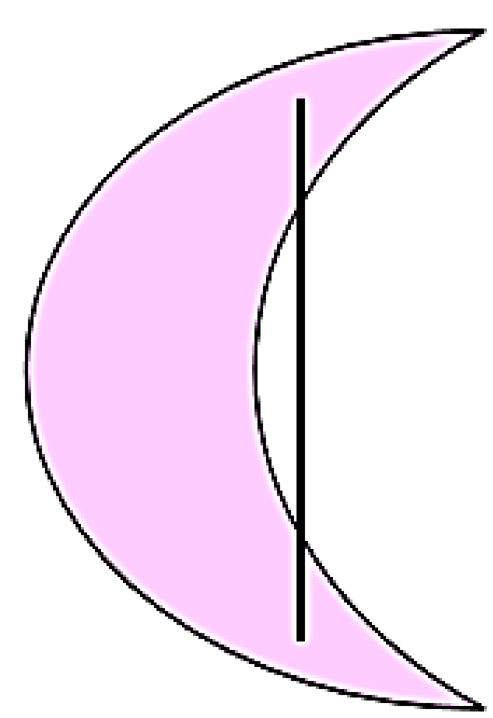
Definition For $x, y \in \mathbb{R}^n$, $\texttt{dist}(x, y) = \sqrt{\sum_{i=1}^{n}(x(i) - y(i))^2}$ denotes the Euclidean distance between points x and y.

A function $f : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}^n$ is **continuous at a point** $x \in D$ if for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $y \in D$: if $\texttt{dist}(x, y) < \delta$ then $\texttt{dist}(f(x), f(y)) < \epsilon$.
f is called **continuous** if it is continuous at every point $x \in D$.

Definition A set $K \subseteq \mathbb{R}^n$ is **convex** if for all $x, y \in K$ and all $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in K$.



convex set



not a convex set

Rather than stating a general definition of compactness for arbitrary topological spaces, we use the following fact as a definition, restricted to Euclidean space:

Fact A set $K \subseteq \mathbb{R}^n$ is **compact** if and only if it is **closed** and **bounded**. (So, we need to define "closed" and "bounded".)

Definition A set $K \subseteq \mathbb{R}^n$ is **bounded** iff there is some non-negative integer M, such that $K \subseteq [-M, M]^n$.
(i.e., K "fits inside" a finite n-dimensional box.)

Definition A set $K \subseteq \mathbb{R}^n$ is **closed** iff for all sequences $x_0, x_1, x_2, \ldots$, where for all $i \geq 0$, $x_i \in K$, if there exists $x \in \mathbb{R}^n$ such that $x = \lim_{i\to\infty} x_i$ (i.e., for all $\epsilon > 0$, there exists integer $k > 0$ such that $\mathtt{dist}(x, x_m) < \epsilon$ for all $m > k$), then $x \in K$.
(In other words, if a sequence of points is in K then its limit (if it exists) must also be in K.)